

# Generalized Burnside Algebra of type $B_n$

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## Abstract

We define the generalized Burnside algebra  $HB(W_n)$  for  $B_n$ -type Coxeter group  $W_n$  and construct an surjective algebra morphism between Mantaci-Reutenauer algebra  $\sum'(W_n)$  and  $HB(W_n)$ . Then, by obtaining the primitive idempotents  $(e_\lambda)_{\lambda \in \mathcal{DP}(n)}$  of  $HB(W_n)$ , we consider the image  $\text{res}_{W_A}^{W_n} e_B$  and  $\text{ind}_{W_A}^{W_n} e_B^A$  under restriction and induction map between generalized Burnside algebras. We give an alternative formula to compute the elements number of conjugate classes of  $W_n$ . We also obtain an effective method to determine the size of  $\mathcal{C}(S_n)$  which is the set of elements of type  $S_n$ .

**Keywords:** Mantaci-Reutenauer Algebra, Generalized Burnside Algebra, Orthogonal Primitive Idempotents

## 1 INTRODUCTION

Let  $W_n$  be the Coxeter group of type  $B_n$ . In this paper, we first introduce the Burnside algebra generated by isomorphism classes of reflection subgroups of  $W_n$  corresponding to signed compositions of  $n$ . It is called *generalized Burnside algebra* of type  $B_n$ . Then we investigate the representation theory of this algebra and its relation to Mantaci-Reutenauer algebra constructed in [11]. In [1], authors have constructed parabolic Burnside algebra by using only standard parabolic subgroups of any finite Coxeter groups. Since the class of the reflection subgroups of  $W_n$  corresponding to signed compositions of  $n$  also contains standard parabolic subgroups, the Burnside algebra we have constructed in Section 3 is more general than parabolic Burnside algebra of type  $B_n$ . We have also constructed the primitive idempotents of generalized Burnside algebra in a parallel manner to [1] and then we have obtained their image under restriction and induction map between generalized Burnside algebras. We follow certain notations in [2] and [3] in denoting some concepts.

## 2 NOTATION, PRELIMINARIES

### 2.1 Hyperoctahedral group.

Let  $(W_n, S_n)$  denote a Coxeter group of type  $B_n$  and write its generating set as  $S_n = \{t, s_1, \dots, s_{n-1}\}$ .  $W_n$  acts by the permutation on the set  $X_n = \{-n, \dots, -1, 1, \dots, n\}$  such that for every  $i \in X_n$ ,  $w(-i) = -w(i)$ . So we have,

$$W_n = \{w \in \text{Perm}(X_n) : \forall i \in X_n, w(-i) = -w(i)\}.$$

If  $J \subset S_n$ , we will denote by  $W_J$  *standard parabolic subgroup* of  $W_n$ , which is generated by  $J$ . A *parabolic subgroup* of  $W_n$  is a subgroup of  $W_n$  conjugate to some  $W_J$ . Let  $t_1 := t$  and  $t_i := s_{i-1}t_{i-1}s_{i-1}$  for each  $i$ ,  $2 \leq i \leq n$ . Put  $T_n := \{t_1, \dots, t_n\}$ . It is well-known that there are the following relations between the elements of  $S_n$  and  $T_n$ :

1.  $t_i^2 = 1, s_j^2 = 1$  for all  $i, j$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n-1$ ;

2.  $ts_1ts_1 = s_1ts_1t$ ;
3.  $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$  ( $1 \leq i \leq n-2$ );
4.  $ts_i = s_it$ ,  $i > 1$ ;
5.  $s_is_j = s_js_i$  for  $|i-j| > 1$ ;
6.  $t_it_j = t_jt_i$  for  $1 \leq i, j \leq n$ .

We denote by  $l : W_n \rightarrow \mathbb{N}$  the length function attached to  $S_n$ . Let  $\mathcal{T}_n$  denote the reflection subgroup of  $W_n$  generated by  $T_n$ . It is also clear that  $\mathcal{T}_n$  is a normal subgroup of  $W_n$ . Now let  $S_{-n} = \{s_1, \dots, s_{n-1}\}$  and let  $W_{-n}$  denote the reflection subgroup of  $W_n$  generated by  $S_{-n}$ , where  $W_{-n}$  is isomorphic to the symmetric group  $\Xi_n$  of degree  $n$ . Thus  $W_n = W_{-n} \ltimes \mathcal{T}_n$ . Therefore, we have  $|W_n| = 2^n \cdot n!$ .

Let  $\{e_1, \dots, e_n\}$  be the canonical basis of the Euclidian space  $\mathbb{R}^n$  over  $\mathbb{R}$ . Let

$$\Psi_n^+ = \{e_i : 1 \leq i \leq n\} \cup \{e_j + \lambda e_i : \lambda \in \{-1, 1\} \text{ and } 1 \leq i < j \leq n\}.$$

Then  $\Psi_n$  is a root system of  $B_n$ -type. Because of this, we get  $t_j = s_{e_j}$ , ( $1 \leq j \leq n$ ) and  $s_i = s_{e_{i+1}-e_i}$ , ( $1 \leq i \leq n-1$ ). The set  $\Pi_n = \{e_1, e_2 - e_1, \dots, e_n - e_{n-1}\}$  is a simple system of  $\Psi_n$ . The generating set  $S_n$  of  $W_n$  is denoted by  $\{s_\alpha : \alpha \in \Pi_n\}$  as a set of simple reflections. To have further information about the Coxeter groups of type  $B_n$ , the readers can see [9] and [10]. We set

$$S'_n = S_n \cup T_n = S_n \cup \{t_1, \dots, t_n\}.$$

A *signed composition* of  $n$  is an expression of  $n$  as a finite sequence  $A = (a_1, \dots, a_k)$  whose each part consists of non-zero integers such that  $\sum_{i=1}^k |a_i| = n$ . We set  $|A| = \sum_{i=1}^k |a_i|$ . If exactly  $k$  parts appear in  $A$ , a signed composition of  $n$ , we say that  $A$  has *k-length*. The number of  $k$  will be denoted by  $lg(A)$ . We write  $\mathcal{SC}(n)$  to denote the set of signed compositions of  $n$ . We also note that  $|\mathcal{SC}(n)| = 2 \cdot 3^{n-1}$ . Let  $A \in \mathcal{SC}(n)$ . We denote by  $lg^+(A)$  and  $lg^-(A)$  the length of positive and negative parts of  $A$ , respectively.

Let  $A = (a_1, \dots, a_k) \in \mathcal{SC}(n)$ .  $A$  is said to be *positive* if  $a_i > 0$  for every  $i \geq 1$ . We say that  $A$  is *parabolic* if  $a_i < 0$  for every  $i \geq 2$ . If  $a_i \geq -1$  for every  $i \geq 1$  then we called  $A$  as *semi-positive*. Let define  $A^+ = (|a_1|, \dots, |a_r|)$ . Then  $A^+$  is a positive signed composition of  $n$ . The set of positive and parabolic signed compositions of  $n$  is denoted by  $\mathcal{SC}^+(n)$  and  $\mathcal{SC}_p(n)$ , respectively. If  $A = (a_1, \dots, a_r) \in \mathcal{SC}(m)$  and  $B = (b_1, \dots, b_s) \in \mathcal{SC}(n)$ , then  $A \sqcup B$  will denote the signed composition  $(a_1, \dots, a_r, b_1, \dots, b_s)$  of  $m+n$ .

A *double partition*  $\mu = (\mu^+, \mu^-)$  of  $n$  consists of a pair of partitions  $\mu^+$  and  $\mu^-$  such that  $|\mu| = |\mu^+| + |\mu^-| = n$ . We denote the set of double partitions of  $n$  by  $\mathcal{DP}(n)$ . For  $\mu = (\mu^+, \mu^-) \in \mathcal{DP}(n)$  we set  $\hat{\mu} := \mu^+ \sqcup \mu^-$ , then  $\hat{\mu} \in \mathcal{SC}(n)$ . Because of this description, the map  $\mathcal{DP}(n) \rightarrow \mathcal{SC}(n)$ ,  $\mu \mapsto \hat{\mu}$  is injective.

Now let  $A \in \mathcal{SC}(n)$ . If  $\mu^+$  (resp.  $\mu^-$ ) is rearrangement of the positive parts (resp. negative parts) of  $A$  in increasing order, then  $\lambda(A) := (\lambda^+, \lambda^-)$  is a double partition of  $n$ . Thus, the map  $\lambda : \mathcal{SC}(n) \rightarrow \mathcal{DP}(n)$  is surjective, and furthermore  $\lambda(\hat{\mu}) = \mu$ .

In [2], Bonnafé ve Hohlweg have constructed some reflection subgroups of  $W_n$  corresponding to signed compositions of  $n$  as an analogue of  $\Xi_n$ . For each  $A = (a_1, \dots, a_k) \in \mathcal{SC}(n)$ , the reflection subgroup  $W_A$  of  $W_n$  is generated by  $S_A$ , which is

$$S_A = \{s_p \in W_n : |a_1| + \dots + |a_{i-1}| + 1 \leq p \leq |a_1| + \dots + |a_i| - 1\} \\ \cup \{t_{|a_1| + \dots + |a_{j-1}| + 1} \in T_n\} \mid a_j > 0\} \subset S'_n$$

By the definition of  $S_A$ , we have the isomorphism  $W_A \cong W_{a_1} \times \cdots \times W_{a_k}$ . By taking into account the definition of the generating set  $S_A$  and the isomorphism  $W_A \cong W_{a_1} \times \cdots \times W_{a_r}$ , if  $a_i > 0$  for  $i$ ,  $1 \leq i \leq r$  then  $\text{rank} W_{a_i} = a_i$  or if  $a_i < 0$  then  $\text{rank} W_{a_i} = |a_i| - 1$ . Therefore, we have

$$\text{rank} W_A = |S_A| = n - \text{lg}^-(A).$$

Because of  $\sum_{i=1}^r |a_i| = n$ , we obtain  $\text{rank} W_A = |S_A| \leq n$ .

Let  $S'_A = S'_n \cap W_A$ ,  $\Psi_A = \{\alpha \in \Psi_n : s_\alpha \in W_A\}$  and  $\Psi_A^+ = \Psi_A \cap \Psi_n^+$ . In addition,  $\Psi_A^+$  is a positive root system of  $\Psi_A$  and  $\Pi_A$  is a simple system of  $\Psi_A$  contained in  $\Psi_A^+$ . Thus  $S_A = \{s_\alpha : \alpha \in \Pi_A\}$  and so by [2]  $(W_A, S_A)$  is a Coxeter system. If  $T_A$  is defined as  $T_n \cap W_A$ , then  $\mathcal{T}_A = \mathcal{T}_n \cap W_A = \langle T_A \rangle$ . Thus, there exists a semidirect decomposition as follows:

$$W_A = \langle W_A \cap S_{-n} \rangle \ltimes \mathcal{T}_A.$$

For  $A, B \in \mathcal{SC}(n)$ , we write  $A \subset B$  if  $W_A \subset W_B$ , where  $\subset$  is a partial ordering relation on  $\mathcal{SC}(n)$ . Note that  $S'_A = S'_B$  if and only if  $A = B$ . Let  $A \in \mathcal{SC}(n)$ . Denote by  $c_A$  a Coxeter element of  $W_A$  in terms of generating set  $S_A$ . For  $B, B' \subset A$ , we write  $B \equiv_A B'$  if  $W_B$  is conjugate to  $W_{B'}$  under  $W_A$ . Also, a necessary and sufficient condition that  $c_B$  and  $c_{B'}$  are conjugate to each other in  $W_A$  is that  $B \equiv_A B'$ . This equivalence is a special case for these kind of reflection subgroups of  $W_n$ , because this statement is not true for every reflection subgroups of  $W_n$ . By [7], although some two reflection subgroup  $R$  and  $R'$  of  $W_n$  contain  $W_n$ -conjugate Coxeter element  $c$  and  $c'$  respectively, they can not be  $W_n$ -conjugate to each other. Every element of  $W_n$  is  $W_n$ -conjugate to  $c_A$  for some  $A \in \mathcal{SC}(n)$ . Bonnafé [3] has showed that for  $A, B \in \mathcal{SC}(n)$ ,  $W_A$  is conjugate to  $W_B$  in  $W_n$  if and only if  $\lambda(A) = \lambda(B)$ . The number of conjugate classes of  $W_n$  is equal to  $|\mathcal{DP}(n)|$  and so we may split up  $W_n$  into  $|\mathcal{DP}(n)|$  equivalence classes.

## 2.2 Mantaci-Reutenauer algebra.

For any  $A \in \mathcal{SC}(n)$ , we set

$$D_A = \{x \in W_n : \forall s \in S_A, l(xs) > l(x)\}.$$

By [2] and [9],  $D_A$  is the distinguished coset representatives of  $W_A$  in  $W_n$  provided that the map  $D_A \times W_A \rightarrow W_n$ ,  $(d, w) \mapsto dw$  is bijective. For  $A, B \in \mathcal{SC}(n)$  such that  $B \subset A$ , the set  $D_B^A = D_B \cap W_A$  is the distinguished coset representatives of  $W_B$  in  $W_A$ . Let

$$d_A = \sum_{w \in D_A} w \in \mathbb{Q}W_n,$$

and let

$$\sum'(W_n) = \bigoplus_{A \in \mathcal{SC}(n)} \mathbb{Q}d_A.$$

For every  $A \in \mathcal{SC}(n)$ ,  $\theta_n : \sum'(W_n) \rightarrow \mathbb{Q}\text{Irr}W_n$  be the unique  $\mathbb{Q}$ -linear map such that  $\theta_n(d_A) = \text{Ind}_{W_A}^{W_n} 1_A$ , where  $\mathbb{Q}\text{Irr}W_n$  and  $1_A$  stands for the algebra of irreducible characters of  $W_n$  and the trivial character of  $W_A$ , respectively. For  $A, B \in \mathcal{SC}(n)$ , we put

$$D_{AB} = D_A^{-1} \cap D_B.$$

For a  $x \in D_{AB}$ , by [2],

$$W_A \cap {}^x W_B = W_{A \cap {}^x B}$$

and  $x$  is the unique element of  $W_A x W_B$  of minimal length.

**Theorem 2.1** (2). *If  $A, B$  are two signed composition of  $n$ , then*

- (a)  $\sum'(W_n)$  is a  $\mathbb{Q}$ -subalgebra of  $\mathbb{Q}W_n$ ;
- (b)  $\theta_n$  is a surjective morphism of  $\mathbb{Q}$ -algebras;
- (c)  $\text{Ker}\theta_n = \bigoplus_{A, B \in \mathcal{SC}(n), A \equiv_n B'} \mathbb{Q}(d_A - d_B)$ ;
- (d)  $\text{Ker}\theta_n$  is the radical of  $\sum'(W_n)$ .

$\sum'(W_n)$  is called *Mantaci-Reutenauer algebra* of  $W_n$ , which includes the classical Solomon algebras of  $W_n$  and  $\Xi_n$ . For a subset  $\mathcal{F}$  of  $\mathcal{SC}(n)$ , we set

$$\sum'_{\mathcal{F}}(W_n) = \bigoplus_{A \in \mathcal{F}} \mathbb{Q}d_A.$$

Bonnafé [3] has introduced the order  $\preceq$  on  $\mathcal{SC}(n)$ , which is more useful than  $\subset$  to determine the structure of the multiplication in  $\sum'(W_n)$  in the following way. For  $A, B \in \mathcal{SC}(n)$ , we write  $A \preceq B$  if and only if either

- $A \subset B$  or
- $A \subset B^+, \lg(A) > \lg(B), \lg^-(A) \geq \lg^-(B)$

satisfies. Let  $\mathcal{F}_{\preceq B} = \{A \in \mathcal{SC}(n) : A \preceq B\}$  and let  $\sum'_{\preceq B}(W_n) = \sum'_{\mathcal{F}_{\preceq B}}(W_n)$ . For  $A$  and  $B$  be two signed composition of  $n$ , we define

$$A \subset_{\lambda} B \Leftrightarrow \lambda(A) \subset \lambda(B).$$

Here the relation  $\subset_{\lambda}$  is an reflexive and transitive on  $\mathcal{SC}(n)$ . In other words,  $A \subset_{\lambda} B$  if and only if  $W_A$  is contained in some conjugate of  $W_B$ . We will write  $A \subsetneq_{\lambda} B$  if  $\lambda(A) \subsetneq \lambda(B)$ . We set  $\mathcal{F}_{\subset_{\lambda} B} = \{A \in \mathcal{SC}(n) : A \subset_{\lambda} B\}$  and as an abbreviation we shall simply  $\sum'_{\subset_{\lambda} B}(W_n)$  for  $\sum'_{\mathcal{F}_{\subset_{\lambda} B}}(W_n)$ . We will use the following lemma proved by Bonnafé [3] in the beginning of construction of generalized Burnside algebra.

**Proposition 2.1** (3). *Let  $A$  and  $B$  be any two signed composition of  $n$ . Then,*

(a) *There is a map  $f_{AB} : D_{AB} \rightarrow \mathcal{SC}(n)$  satisfying the following conditions:*

- For every  $x \in D_{AB}$ ,  $f_{AB}(x) \subset B$  and  $f_{AB}(x) \equiv_B x^{-1}A \cap B$ .
- $d_A d_B - \sum_{x \in D_{AB}} d_{f_{AB}(x)} \in \sum'_{\subset_{\lambda} A}(W_n) \cap \sum'_{\prec B}(W_n) \cap \text{Ker}\theta_n$ .

(b) *If  $A$  parabolic or  $B$  is semi-positive, then  $f_{AB}(x) = x^{-1}A \cap B$  for  $x \in D_{AB}$  and  $d_A d_B = \sum_{x \in D_{AB}} d_{x^{-1}A \cap B}$ .*

For  $\lambda \in \mathcal{DP}(n)$  we set  $\tau_{\lambda} : \sum'(W_n) \rightarrow \mathbb{Q}$ ,  $x \mapsto \theta_n(x)(c_{\lambda})$ . The map  $\tau_{\lambda}$  is independent of the choice of  $c_{\lambda} \in C(\lambda)$  and it is also an algebra morphism. Moreover, the set  $\{\tau_{\lambda} : \lambda \in \mathcal{DP}(n)\}$  is the collection of irreducible representations of  $\sum'(W_n)$ .

Throughout this paper, we will use the following facts proved in [3] frequently .

**Proposition 2.2** (3). *For  $A, B \in \mathcal{SC}(n)$ , let define the sets  $D_{AB}^{\subset} = \{x \in D_{AB} : x^{-1}W_A \subset W_B\}$  and  $D_{AB}^{\equiv} = \{x \in D_{AB} : W_A =^x W_B\}$ . Then the following statements hold:*

1.  $\tau_{\lambda(A)}(d_B) = |D_{AB}^{\subset}|$ .

2.  $D_{AB}^{\equiv} = \{x \in W_n : \Pi_A =^x \Pi_B\}.$
3.  $\mathcal{W}(B) = \{w \in W_n : w(\Pi_B) = \Pi_B\};$
4.  $\mathcal{W}(B)$  is a subgroup of  $N_{W_n}(W_B);$
5.  $N_{W_n}(W_B) = \mathcal{W}(B) \ltimes W_B.$

If  $A \equiv B$ , then it is clear  $D_{AB}^{\equiv} = D_{AB}^{\subset}$ . For  $A \in \mathcal{SC}(n)$  we set  $\mathcal{W}(A) = D_{AA}^{\subset}$ .

### 3 Generalized Burnside Algebra of $W_n$

Let  $A, B$  be any two signed composition of  $n$ . By [2], we have that

$$A \equiv_n B \Leftrightarrow W_A \sim W_B \Leftrightarrow [W/W_A] = [W/W_B]$$

where  $[W/W_A]$  represents the isomorphism class of  $W_n$ -set  $W/W_A$ . The orbits of  $W_n$  on  $W/W_A \times W/W_B$  are of the form  $(W_A x, W_B)$  where  $x \in D_{AB}$ . The stabilizer of  $(W_A x, W_B)$  in  $W_n$  is  ${}^{x^{-1}}W_A \cap W_B = W_{x^{-1}A \cap B}$ . Therefore

$$[W/W_A] \cdot [W/W_B] = [W/W_A \times W/W_B] = \sum_{x \in D_{AB}} [W/W_{x^{-1}A \cap B}].$$

Thus, we are now in a position to give the following definition.

**Definition 3.1.** *The generalized Burnside algebra of  $W_n$  is  $\mathbb{Q}$ -spanned by the set  $\{[W/W_A] : A \in \mathcal{SC}(n)\}$  and it is denoted by  $HB(W_n)$ .*

From Proposition 2.1 and the structure of  $\text{Ker}(\theta_n)$ , the multiplication in  $\sum'(W_n)$  may be expressed by

$$d_A d_B = \sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum_{N \equiv_n N'} a_{NN'}(d_N - d_{N'}),$$

where  $a_{NN'} \in \mathbb{Z}$ ;  $N, N' \subsetneq_{\lambda} A$ ;  $N, N' \prec B$ ;  $f_{AB}(x) \subset B$  and  $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B$ .

Now we define

$$\psi : \sum'(W_n) \rightarrow HB(W_n), \quad d_A \mapsto [W/W_A].$$

the map  $\psi$  is well-defined and surjective. On considering the structure of  $\text{Ker}\theta_n$  and  $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B \Rightarrow W_{f_{AB}(x)} \sim_{W_B} W_{x^{-1}A \cap B}$ , we get

$$\begin{aligned} \psi(d_A d_B) &= \psi\left(\sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum_{N \equiv_n N'} a_{NN'}(d_N - d_{N'})\right) \\ &= \sum_{x \in D_{AB}} \psi(d_{f_{AB}(x)}) + \sum_{N \equiv_n N'} a_{NN'}(\psi(d_N) - \psi(d_{N'})) \\ &= \sum_{x \in D_{AB}} [W/W_{f_{AB}(x)}] \\ &= \sum_{x \in D_{AB}} [W/W_{x^{-1}A \cap B}] \\ &= [W/W_A] \cdot [W/W_B] \\ &= \psi(d_A) \psi(d_B). \end{aligned}$$

Then the map  $\psi$  is an algebra morphism. Since  $\dim_{\mathbb{Q}} \text{HB}(W_n) = \dim_{\mathbb{Q}} \text{QIrr} W_n = |\mathcal{DP}(n)|$ , then there is an algebra isomorphism between  $\text{QHB}(W_n)$  and  $\text{QIrr} W_n$  such that

$$\text{HB}(W_n) \rightarrow \text{QIrr} W_n, [W/W_A] \mapsto \text{ind}_{W_A}^{W_n} 1_A.$$

Now let  $\lambda, \mu \in \mathcal{DP}(n)$  and let  $\varphi_\lambda = \text{ind}_{W_A}^{W_n} 1_A$  for any  $A \in \lambda^{-1}(\lambda)$ . By [2],  $\varphi_\lambda(c_\lambda) = \pi_\lambda(x_{\hat{\lambda}}) = |D_{\hat{\lambda}\hat{\lambda}}^\subset| \neq 0$  and  $\pi_\lambda(x_{\hat{\mu}}) = 0$  if  $\lambda \not\subseteq \mu$ . Thus the matrices  $(\pi_\lambda(c_{\hat{\mu}}))_{\lambda, \mu \in \mathcal{DP}(n)}$  is lower diagonal. Then  $(\varphi_\lambda(c_\mu))_{\lambda, \mu}$  is upper diagonal and has also positive diagonal entries. Therefore  $(\varphi_\lambda(c_\mu))_{\lambda, \mu}$  is invertible and its inverse is  $(u_{\lambda\mu})_{\lambda, \mu \in \mathcal{DP}(n)}$ . As in [1], we define

$$e_\lambda = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \varphi_\mu. \quad (3.1)$$

By definition of  $e_\lambda$  and  $(\varphi_\lambda(c_\mu))^{-1} = (u_{\lambda\mu})$ , we obtain that

$$e_\lambda(c_\mu) = \sum_{\gamma \in \mathcal{DP}(n)} u_{\lambda\gamma} \varphi_\gamma(c_\mu) = \delta_{\lambda, \mu}.$$

Hence the set  $\{e_\lambda : \lambda \in \mathcal{DP}(n)\}$  is a collection of primitive idempotents of  $\text{HB}(W_n)$ . Wedderburn's structure theorem which we need in our works is given as follows:

**Theorem 3.1** (5). *Let  $A$  be an algebra over the field  $K$  and let  $J(A)$  be the Jacobson radical of  $A$ . Then we have*

$$A/J(A) \cong \oplus_{i=1}^s A_i,$$

where  $A_i$  is an matrix algebra isomorphic to  $\text{Mat}_{n_i}(K)$ .

**Lemma 3.1** (5). *Let  $A$  be an commutative algebra over the field  $K$ . Every algebra map  $A \rightarrow K$  is of the form  $s_i$  such that  $s_i(1_{A_j}) = \delta_{i,j}$ , where the  $1_{A_i}$ 's are primitive idempotents.*

For each  $A \in \mathcal{SC}(n)$  by [6],

$$s_A : \text{HB}(W_n) \rightarrow \mathbb{Q}, s_A([X]) = |{}^W A X|$$

is an algebra map, where  ${}^W A X = \{x \in X : W_A x = x\}$ . Since  $\text{HB}(W_n)$  is semisimple and commutative algebra, then all the maps  $\text{HB}(W_n) \rightarrow \mathbb{Q}$  are of the form  $s_A$  for every  $A \in \mathcal{SC}(n)$ . Consequently, we have  $\text{HB}(W_n) = \oplus_{\lambda \in \mathcal{DP}(n)} \mathbb{Q} e_\lambda$ . The proof of the following lemma is immediately seen from [6].

**Lemma 3.2.** *For  $A, B \in \mathcal{SC}(n)$ , we have that*

$$s_A = s_B \Leftrightarrow \lambda(A) = \lambda(B).$$

The dual basis of  $\text{HB}(W_n)$  is  $\{s_{\lambda(A)} : \lambda \in \mathcal{DP}(n)\}$ . For any  $\lambda, \mu \in \mathcal{DP}(n)$ , there exists the following equality

$$s_\lambda(e_\mu) = \delta_{\lambda, \mu}, \quad (3.2)$$

and so any element  $x$  in  $\text{HB}(W_n)$  can be expressed as  $x = \sum_{\lambda \in \mathcal{DP}(n)} s_\lambda(x) e_\lambda$ .

Let  $A$  be a signed composition of  $n$ . Induction and restriction of characters give rise to two maps between  $\text{HB}(W_A)$  and  $\text{HB}(W_n)$ . For any  $A, B \in \mathcal{SC}(n)$  such that  $B \subset A$ , we have  $\text{ind}_{W_A}^{W_n}([W_A/W_B]) = [W_n/W_B]$ .

**Definition 3.2.** Let be  $A, B \in \mathcal{SC}(n)$  such that  $B \subset A$ . The *restriction* is a linear map

$$\text{res}_{W_B}^{W_A} : \text{HB}(W_A) \rightarrow \text{HB}(W_B), \text{res}_{W_B}^{W_A}([W_A/W_C]) = \sum_{d \in W_A \cap D_{CB}} [W_B/W_{B \cap d^{-1}C}].$$

Before going into a further discussion of the restriction and induced character theories of generalize Burnside algebra, we shall first mention the number of elements of the conjugacy class of  $W_A$  in  $W_n$ .

Let  $A, B \in \mathcal{SC}(n)$ . We define

$$\text{inv}_{(W_n/W_A)}(W_B) = \{xW_A : x \in W_n, W_B x W_A = xW_A\},$$

which is the set of elements in  $W_n/W_A$  fixed by  $W_B$ . Also  $\text{inv}_{W_n/W_A}(W_B) = \{xW_A : x \in W_n, x^{-1}W_B x \leq W_A\}$  and so  $\text{inv}_{W_n/W_A}(W_B) = \emptyset$  unless  $\lambda(B) \not\subset \lambda(A)$ . The *mark* of  $W_B$  on  $W_n/W_A$  is defined to be the number

$$\mu_{AB} = |\text{inv}_{W_n/W_A}(W_B)|.$$

From [6], for each  $A' \in \lambda(A)$  and  $B' \in \lambda(B)$ , it is clear  $\mu_{A'B'} = \mu_{AB}$ .

**Definition 3.3.** The *signed table of marks* of  $W_n$  is a square matrix of type  $|\mathcal{DP}(n)| \times |\mathcal{DP}(n)|$  as follows

$$M(W_n) = (\mu_{\lambda\vartheta})_{\lambda, \vartheta \in \mathcal{DP}(n)}.$$

This table also contains the parabolic table of marks of  $W_n$ .  $M(W_n)$  is lower triangular with non-zero diagonal entries and so it is invertible. Since  $|D_{BA}^\subset|$  is the number of fixed points in  $W_n/W_A$  under the action of  $c_B$ , we have

$$\mu_{AB} = |\text{inv}_{W_n/W_A}(W_B)| = |D_{BA}^\subset| = \pi_{\lambda(B)}(d_A).$$

As a corollary, each entries of the matrix  $M(W_n)$  can also be read from the matrices  $(\pi_\lambda(c_{\hat{\mu}}))_{\lambda, \mu \in \mathcal{DP}(n)}$ .

**Proposition 3.1.** Let  $A, B \in \mathcal{SC}(n)$ . Then we have

$$\mu_{AB} = |\text{inv}_{W_n/W_A}(W_B)| = |\{xW_A : x \in W_n, W_B \leq^x W_A\}| \cdot |\mathcal{W}(A)|.$$

The proof is an easy application of the definition of  $\mu_{AB}$ . As a result of the above proposition, the number of conjugates of  $W_A$  containing  $W_B$  is

$$|\{xW_A : x \in W_n, W_B \leq^x W_A\}| = \frac{|D_{BA}^\subset|}{|\mathcal{W}(A)|}.$$

**Proposition 3.2.** Let  $A \in \mathcal{SC}(n)$  and  $\lambda(A) = \lambda$ . The number of all reflection subgroups of  $W_n$  that are conjugate to  $W_A$  is

$$|[W_A]| = |D_A| \cdot u_{\lambda, \lambda}.$$

**Proof:** Put  $[W_A] = \{xW_A : x \in W_n\}$ . Now we note that  $xW_A x^{-1} = yW_A y^{-1}$  if and only if  $x \in yN_{W_n}(W_A)$ . Thus, the number of distinct conjugates of  $W_A$  in  $W_n$  is  $[W_n : N_{W_n}(W_A)]$ . Since also  $N_{W_n}(W_A) = \mathcal{W}(A) \rtimes W_A$ , we have

$$|[W_A]| = \frac{|W_n|}{|\mathcal{W}(A)| \cdot |W_A|} = \frac{|D_A|}{|\mathcal{W}(A)|}.$$

Likewise, from the fact that  $\pi_{\lambda(A)}(d_A) = |D_{AA}^\subset| = |\mathcal{W}(A)|$  and  $\varphi_\lambda(\text{cox}_\lambda) = \pi_{\lambda(A)}(d_A) = \frac{1}{u_{\lambda, \lambda}}$ , we see that  $[|W_A|]$  has the desired number.  $\square$

**Example 3.1.** The set  $D_{(2,1)} = \{1, s_2, s_1 s_2\}$  is the distinguished coset representatives of reflection subgroup  $W_{(2,1)}$  in  $W_3$ , then the number of all reflection subgroups conjugate to  $W_{(2,1)}$  in  $W_3$  is

$$|[W_{(2,1)}]| = |D_{(2,1)}| \cdot u_{(2,1),(2,1)} = 3 \cdot 1 = 3.$$

These are explicitly  $W_{(2,1)}$ ,  $W_{(1,2)}$  and  ${}^{s_2}W_{(2,1)} = \langle s_2 s_1 s_2, t_1, t_2 \rangle$ . We note that the last one is not a reflection subgroup of  $W_3$  corresponding to any signed composition of 3.

**Remark 3.1.** For  $A, B \in \mathcal{SC}(n)$  such that  $B \subset A$  and for any  $x \in HB(W_n)$ , by using the definition of  $s_A$  one can see that there is the relation  $s_B^A(\text{res}_{W_A}^{W_n}(x)) = s_B(x)$ .

We can now give the following proposition which is analog to Theorem 3.2.4 in [1].

**Proposition 3.3.** Let  $A, B \in \mathcal{SC}(n)$  and let  $A_1, A_2, \dots, A_r$  be representatives of the  $W_A$ -equivalent classes of subsets of  $A$ , which is  $W_n$ -equivalent to  $B$ . Then,

$$\text{res}_{W_A}^{W_n} e_B = \sum_{i=1}^r e_{B_i}^A.$$

If  $B$  is not  $W_n$ -equivalent to any subset of  $A$  then  $\text{res}_{W_A}^{W_n} e_B = 0$ .

**Proof:** Since  $\text{res}_{W_A}^{W_n} e_B$  is an element of  $\mathbb{Q}HB(W_A)$  we have  $\text{res}_{W_A}^{W_n} e_B = \sum_{A_i \subset A} s_{B_i}^A(\text{res}_{W_A}^{W_n}(e_B)) \xi_{C_i}^C$ . Then by using Remark (3.1) and the relation (3.2), we get

$$\begin{aligned} \text{res}_{W_A}^{W_n} e_B &= \sum_{A_i \subset A} s_{A_i}(e_B) e_{A_i}^A \\ &= \sum_{\substack{A_i \subset A \\ A_i \equiv_A B}} e_{A_i}^A \\ &= \sum_{i=1}^r e_{B_i}^A. \end{aligned}$$

□

**Proposition 3.4.** Let  $A, B \in \mathcal{SC}(n)$  and let  $B \subset A$ . Then we have

$$\text{ind}_{W_A}^{W_n} e_B^A = \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B)|} \cdot e_B.$$

**Proof:** Firstly, we assume that  $A = B$  and  $c_A$  is a Coxeter element of  $W_A$ . Since the image of  $c_A$  under permutation character of  $W_n$  on the cosets of  $W_A$  is  $|\mathcal{W}(A)|$  then it follows from the fact that

$$x^{-1} c_A x \in W_A \Leftrightarrow x \in N_{W_n}(W_A).$$

Thus we obtain

$$\begin{aligned} \text{ind}_{W_A}^{W_n} e_A^A(c_A) &= |D_A \cap N_{W_n}(W_A)| \\ &= |\mathcal{W}(A)|. \end{aligned}$$

As  $\text{ind}_{W_A}^{W_n} e_A^A$  takes value zero except for the elements conjugate to  $c_A$  and so we get

$$\text{ind}_{W_A}^{W_n} e_A^A = |\mathcal{W}(A)| e_A.$$



By transitivity of induced characters, we generally get

$$\begin{aligned}\text{ind}_{W_A}^{W_n} e_B^A &= \text{ind}_{W_A}^{W_n} \left( \frac{1}{|W_A \cap \mathcal{W}(B)|} |W_A \cap \mathcal{W}(B)| e_B^A \right) \\ &= \text{ind}_{W_A}^{W_n} \left( \frac{1}{|W_A \cap \mathcal{W}(B)|} \text{ind}_{W_B}^{W_A} e_B^B \right) \\ &= \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B)|} e_B.\end{aligned}$$

□

Furthermore, there is also the equality  $\text{ind}_{W_A}^{W_n} e_B^A = |N_{W_n}(W_B) : N_{W_A}(W_B)| e_B$ .

**Theorem 3.2.** *Let  $A, B \in \mathcal{SC}(n)$  and let  $\lambda(B) \subset \lambda(A)$ . If  $B_1, B_2, \dots, B_r$  are the representatives of the  $W_A$ -equivalent classes of subsets of  $A$ ,  $W_n$ -equivalent to  $B$ , then for  $c_B \in W_n$ ,*

$$\text{ind}_{W_A}^{W_n} 1_A(c_B) = \sum_{i=1}^r \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B_i)|}.$$

**Proof:** Let  $A, B \in \mathcal{SC}(n)$ . If  $A \equiv_n B$  then it is easy to prove that  $|\mathcal{W}(A)| = |\mathcal{W}(B)|$ .  $1_A = \sum_E e_E^A$ , where  $E \in \mathcal{SC}(n)$  runs over  $W_A$ -conjugate classes of subsets of  $A$ . From Proposition 9, we have

$$\text{ind}_{W_A}^{W_n} 1_A = \sum_E \text{ind}_{W_A}^{W_n} e_E^A \Rightarrow \text{ind}_{W_A}^{W_n} 1_A = \sum_E \frac{|\mathcal{W}(E)|}{|W_A \cap \mathcal{W}(E)|} \cdot e_E.$$

As each  $B_i$  is  $W_n$ -congruent to  $B$ , then  $e_E(c_B) = 1$  if and only if  $E \equiv_{W_A} B_i$ . Thus from (3.1), we obtain that

$$\text{ind}_{W_A}^{W_n} 1_A(c_B) = \sum_{i=1}^r \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B_i)|}.$$

Hence the theorem is proved. □

Theorem 3.3 and Proposition 3.5 give us a useful computation to determine the coefficient of the sign character  $\varepsilon_n$  in the expression of primitive idempotent  $e_\lambda$ ,  $\lambda \in \mathcal{DP}(n)$  in terms of irreducible characters of  $W_n$ . From now on, for convention, we simply write  $e_\lambda$  instead of  $e_\lambda$  to avoid the complexity of parenthesis in some calculation.

**Theorem 3.3.**  $u_{(n),(-1,\dots,-1)} = \frac{(-1)^n}{2n}$ .

**Proof:** Let  $\varepsilon_n : W_n \rightarrow \{-1, 1\}$  and  $\chi_{reg} : W_n \rightarrow \mathbb{Z}$  be the sign and regular character of  $W_n$ , respectively. For  $A = (-1, \dots, -1)$  it is satisfied  $\text{ind}_{W_A}^{W_n} 1_A = \chi_{reg}$ . The character  $\varepsilon_n$  is contained in  $\chi_{reg}$  with the property that its coefficient is just 1, thus we have

$$\langle \text{ind}_{W_A}^{W_n} 1_A, \varepsilon_n \rangle = 1.$$

Now let  $A \neq (-1, \dots, -1)$ . By Frobenius Reciprocity and  $\sum_{w \in W_A} (-1)^{l(w)} = 0$ , it is obtained that  $\langle \text{ind}_{W_A}^{W_n} 1_A, \varepsilon_n \rangle = 0$ . If  $w$  is conjugate to  $c_{W_n}$ , then  $e_{(n)}(w) = 1$  and  $l(w) = n$ . Let  $ccl_{W_n}(c_{W_n})$  denotes the conjugate class of  $c_{W_n}$  in  $W_n$ . By [4], by considering the fact  $|ccl_{W_n}(c_{W_n})| = \frac{|W_n| \cdot n}{2N}$ , we have

$$\langle e_{(n)}, \varepsilon_n \rangle = \frac{(-1)^n}{2n}.$$

On the other hand,  $\langle e_{(n)}, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{(n)\mu} \langle \varphi_\mu, \varepsilon_n \rangle = u_{(n),(-1,\dots,-1)}$  and so the proof is completed. □

**Proposition 3.5.** For  $\lambda \in \mathcal{DP}(n)$  and  $\lambda \neq (n)$ , then we have

$$u_{\lambda,(-1,\dots,-1)} = (-1)^{|S_{\lambda}|} \cdot \frac{|\mathcal{C}(\lambda)|}{|W_n|}.$$

**Proof:** Since the sign character is constant on conjugate classes, then we have

$$\begin{aligned} \langle e_{\lambda}, \varepsilon_n \rangle &= \frac{1}{|W_n|} \sum_{w \in \mathcal{C}(\lambda)} (-1)^{l(w)} \\ &= (-1)^{|S_{\lambda}|} \cdot \frac{|\mathcal{C}(\lambda)|}{|W_n|}. \end{aligned}$$

Note that  $\langle \varphi_{\mu}, \varepsilon_n \rangle$  has value 1 for  $\mu = (-1, \dots, -1)$  and zero for the others. Thereby, we obtain  $\langle e_{\lambda}, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \langle \varphi_{\mu}, \varepsilon_n \rangle = u_{\lambda,(-1,\dots,-1)}$ . Eventually, we have  $u_{\lambda,(-1,\dots,-1)} = (-1)^{|S_{\lambda}|} \cdot \frac{|\mathcal{C}(\lambda)|}{|W_n|}$ .  $\square$

Notice that calculation of the inner product  $\langle e_{\lambda}, 1_{W_n} \rangle$  leads to the following corollary.

**Corollary 3.1.** Let  $\lambda \in \mathcal{DP}(n)$ . Then

$$|W_n| \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda,\mu} = |\mathcal{C}(\lambda)|.$$

By means of this Corollary and the matrices  $(u_{\lambda\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$ , we can readily determine the size of the conjugacy classes of  $W_n$ . The proof of the next lemma is an immediate consequence of the inner product of characters.

For a subset  $X$  of  $W_n$ , we denote by the subspace  $\text{Fix}(X) = \{v \in \mathbb{R}^n : \forall x \in X \ x(v) = v\}$  of  $\mathcal{R}^n$  fixed by  $X$  and let write  $W_{\text{Fix}(X)} = \{w \in W_n : \forall v \in \text{Fix}(X) \ w(v) = v\}$  for the stabilizer of  $\text{Fix}(X)$  in  $W_n$ . By [7], the set  $W_{\text{Fix}(X)}$  is called the *parabolic closure* of  $X$  and it is denoted by  $A(X)$ . For a  $w \in W_n$ , if we take  $X = \{w\}$  then we write  $\text{Fix}(w)$  and  $A(w)$  instead of  $\text{Fix}(\{w\})$  and  $A(\{w\})$ , respectively. By [1], if  $A(w)$  is  $W_n$ -conjugate to  $W_J$  for some  $J \subset S_n$ , then we say that  $w$  is of *type*  $J$ .

**Proposition 3.6.** If  $B \in \mathcal{SC}^+(n)$ , then we have  $A(W_B) = W_n$ .

**Proof:** Since  $B \in \mathcal{SC}^+(n)$ , we have  $\mathcal{T}_n \leq W_B$  and so  $w_n \in W_B$ . By considering  $w_n$  as a linear map  $-id_{\mathbb{R}^n}$ , we obtain  $\text{Fix}(w_n) = \{\vec{0}\}$ . Thus, the parabolic closure of  $w_n$  is  $A(w_n) = W_{\text{Fix}(w_n)} = W_n$ . Because of the relation  $w_n \in W_B \subset A(c_B) = A(W_B)$ , we get  $w_n \in A(c_B)$ . By [12], the inclusion  $A(w_n) \subset A(c_B) = A(W_B)$  holds. If we take into account the fact that  $A(w_n) = W_n$ , then we have  $A(W_B) = W_n$ . This completes the proof.  $\square$

As a consequence of Proposition 3.6, if  $B \in \mathcal{SC}^+(n)$ , then the parabolic closure of  $W_B$  is  $W_n$  and each element of  $\mathcal{C}(\lambda(B))$  is of type  $S_n$ .

**Lemma 3.3.** Let  $A$  be a signed composition of  $n$ . Then  $w_n$  belongs to  $W_A$  if and only if  $A \in \mathcal{SC}^+(n)$ .

**Proof:** When  $A$  is a positive signed composition of  $n$ , we can easily see from the proof of Proposition 3.6 that  $w_n$  is an element of  $W_A$ . Conversely, let  $w_n$  be in  $W_A$ . We suppose that  $A = (a_1, \dots, a_i, \dots, a_r)$  is not a positive signed composition of  $n$ . Then there exists  $a_i < 0$  for some  $i$ ,  $1 \leq i \leq r$ . Thus from the definition of  $W_A$ , we obtain  $t_{|a_1|+\dots+|a_i|} \notin S'_A$ . Hence for  $x \in W_A$  and  $e_{|a_1|+\dots+|a_i|} \in \mathbb{R}^n$ , we have  $x(e_{|a_1|+\dots+|a_i|}) = e_{|a_1|+\dots+|a_i|}$  and so  $e_{|a_1|+\dots+|a_i|} \in \text{Fix}(W_A)$ . This is a contradiction, because from Proposition 3.6 the subspace  $\text{Fix}(W_A)$  consists only  $\vec{0}$ . Therefore, we have  $A \in \mathcal{SC}^+(n)$ .  $\square$

**Proposition 3.7.** *If the set  $\mathcal{C}(S_n)$  denotes the set of all elements of type  $S_n$ , then we have*

$$\mathcal{C}(S_n) = \coprod_{A \in \mathcal{SC}^+(n)} \mathcal{C}(\lambda(A)). \quad (3.3)$$

**Proof:** From Proposition 3.6, for all  $A \in \mathcal{SC}^+(n)$  every element of  $\mathcal{C}(\lambda(A))$  is of type  $S_n$  and so the reverse inclusion holds. Now let  $w \in \mathcal{C}(S_n)$ . Then  $w$  is  $W_n$ -conjugate to  $c_A$  for some  $A \in \mathcal{SC}(n)$ . Thus we get  $A(w) = A(c_A) = A(W_A) = W_n$ . From here, for every  $x \in W_n$  and every  $v \in \text{Fix}(W_A)$  we obtain  $x(v) = v$ . In particular, if we take  $w_n = -id_{\mathcal{R}^n} \in W_n$ , then it is seen that  $\text{Fix}(W_A)$  includes just  $\{\vec{0}\}$ . Thus  $w_n$  is an element of  $W_A$ . Otherwise, if  $A \notin \mathcal{SC}^+(n)$ , then from the proof of Lemma 3.3 we get  $\text{Fix}(W_A) \neq \{\vec{0}\}$ , which is a contradiction. Hence  $A \in \mathcal{SC}^+(n)$ . Thus  $w$  belongs to  $\mathcal{C}(\lambda(A))$  and so it is seen that the inclusion  $\mathcal{C}(S_n) \subset \coprod_{A \in \mathcal{SC}^+(n)} \mathcal{C}(\lambda(A))$  satisfies. It is required.  $\square$

Since the exponents of  $W_n$  are in turn  $1, 3, \dots, 2n-1$ , then from [1] the number of elements of  $S_n$ -type is equal to the product of exponents of  $W_n$  and so  $|\mathcal{C}(S_n)| = 1 \cdot 3 \cdots 2n-1$ . By the equality (3.3), we obtain the formula

$$|\mathcal{C}(S_n)| = \sum_{A \in \mathcal{SC}^+(n)} |\mathcal{C}(\lambda(A))|. \quad (3.4)$$

Thus Proposition 3.7 gives us an alternative method to compute the number of elements of type  $S_n$ .

**Lemma 3.4.** *Let  $\varepsilon_n$  be the sign character of  $W_n$  and let  $A \in \mathcal{SC}(n)$ . Then  $\text{res}_{W_A}^{W_n} \varepsilon_n$  is an irreducible character of  $W_A$ .*

Let  $A \in \mathcal{SC}(n)$ . When  $A$  is not a parabolic signed composition of  $n$ , then  $l_A(w)$  may not equal to  $l(w)$  for some  $w \in W_A$ . The following lemma gives an interesting relation between these length functions.

**Lemma 3.5.** *Let  $A \in \mathcal{SC}(n)$ . Let  $l$  be the length function of  $W_n$  attached to  $S_n$  and let  $l_A$  be the length function of  $W_A$  in relation to  $S_A$ . Then for every  $w \in W_A$*

$$l(w) \equiv l_A(w) \pmod{2}.$$

**Proof:** We know that  $t_1 := t$  and  $t_i := s_{i-1}t_{i-1}s_{i-1}$  for each  $i$ ,  $2 \leq i \leq n$ . It is immediately seen that  $l(t_i) = 2i - 1$  according to the generating set  $S_n$ . Because  $s_{i-1}s_{i-2} \cdots t_1 \cdots s_{i-2}s_{i-1}$  is a reduced expression for  $t_i$ , then  $l(t_i)$  is equal to  $2i - 1$ . Therefore for every  $i$ ,  $1 \leq i \leq n$ ,  $l(t_i) \equiv 1 \pmod{2}$ . Let  $w \in W_A$ . By virtue of  $S_A \subset S'_n$ , a reduced expression of  $w$  in terms of the elements of  $S_A$  can also include some elements of  $T_n$  besides those of  $S_n$ . The length of  $t_i$  which may be included in this reduced expression is 1 under  $l_A$ , and also  $2i - 1$  in terms of  $l$ . Thus, the length of  $t_i$  is odd number with respect to both length functions. However, since the length of  $s_i$  according to both  $l_C$  and  $l$  is 1, then the statement of being odd or even for  $w$  is the same in terms of both length functions. If we denote by  $N_A(w)$  and  $N(w)$  the sets of positive roots in  $\Psi_A^+$  and  $\Psi^+$  transformed into negative roots by  $w$ , respectively, then it follows that

$$N_A(w) \subset N(w).$$

It is seen from this fact that the reduced expression of  $w \in W_A$  in terms of the fundamental reflections in  $S_A$  is not longer than relative to those of  $S_n$ . Let  $A = (a_1, \dots, a_r)$ . Moreover, then because of  $W_A \cong W_{a_1} \times \cdots \times W_{a_r}$ , expanding the reduced expression of  $w \in W_A$  according to simple reflections of  $S_A$  by using the relations  $t_i = s_{i-1}s_{i-2} \cdots t_1 \cdots s_{i-2}s_{i-1}$  yield that reduced

expression of  $w$  in terms of simple reflections in  $S_n$  can be obtained without any simplification. Hence  $l(w) \equiv l_C(w) \pmod{2}$ , as required.  $\square$

As a result of the previous lemma, we get

$$\varepsilon_n(w) = (-1)^{l(w)} = (-1)^{l_A(w)} = \varepsilon_A(w).$$

Thus from Lemma 4 and 5, for every  $A \in \mathcal{SC}(n)$  it is obtained that  $\text{res}_{W_A}^{W_n} \varepsilon_n = \varepsilon_A$ .

**Example 3.2.** For a concrete example, let  $A = (-2, 3, -1, -3, 1) \in \mathcal{SC}(10)$ ,  $S_A = \{s_1\} \cup \{t_3, s_3, s_4\} \cup \{s_7, s_8\} \cup \{t_{10}\} \subset S'_{10}$  and  $S_A = \{s_1\} \cup \{t_3, s_3, s_4, t_4, t_5\} \cup \{s_7, s_8\} \cup \{t_{10}\}$ . Then  $W_A \cong W_2 \times W_3 \times W_1 \times W_3 \times W_1$ . For  $w = s_7 t_3 s_3 s_1 t_{10} \in W_A$ ,  $l_A(w) = 5$  and also

$$w = s_7 t_3 s_3 s_1 t_{10} = s_7 s_2 s_1 t_1 s_1 s_2 s_3 s_1 s_9 s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_1 t_1 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 \in W_n,$$

so  $l(w) = 27$ . It follows that  $l(w) \equiv l_A(w) \equiv 1 \pmod{2}$ .

**Theorem 3.4.** Let  $A \in \mathcal{SC}(n)$  and  $\lambda \in \mathcal{DP}(n)$ . Then

$$\sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} a_{\hat{\mu}A(-1, \dots, -1)} = (-1)^{|S_\lambda|} \frac{|\mathcal{C}(\lambda) \cap W_A|}{|W_A|},$$

where  $a_{\hat{\mu}A(-1, \dots, -1)} = |\{x \in D_{\hat{\mu}A} : x^{-1} \hat{\mu} \cap A = (-1, \dots, -1)\}|$ .

**Proof:** The term  $d_{(-1, \dots, -1)}$  in the multiplication  $d_{\hat{\mu}} d_A$  lies in the summand  $\sum_{x \in D_{\hat{\mu}A}} d_{f_{\hat{\mu}A}(x)}$  from the structure of  $\text{Ker} \theta_n$  and Proposition 2.1 (a). If we write the coefficient of  $d_{(-1, \dots, -1)}$  in this summand as  $a_{\hat{\mu}A(-1, \dots, -1)}$ , and so we get

$$a_{\hat{\mu}A(-1, \dots, -1)} = |\{x \in D_{\hat{\mu}A} : f_{\hat{\mu}A}(x) = (-1, \dots, -1)\}|.$$

By using Proposition 2.1 (a) along with the fact  $f_{\hat{\mu}A}(x) \equiv_A x^{-1} \hat{\mu} \cap A$ , it is seen that there is the equivalence  $x^{-1} \hat{\mu} \cap A \equiv_A (-1, \dots, -1)$ . Since no elements in  $\mathcal{SC}(n)$  is congruent to  $(-1, \dots, -1)$  except for  $(-1, \dots, -1)$ , it then follows that  $x^{-1} \hat{\mu} \cap A = (-1, \dots, -1)$ . Hence we have deduced the equality  $a_{\hat{\mu}A(-1, \dots, -1)} = |\{x \in D_{\hat{\mu}A} : x^{-1} \hat{\mu} \cap A = (-1, \dots, -1)\}|$  holds. Therefore, by Frobenius Reciprocity and Mackey Theorem, we have

$$\begin{aligned} \langle \xi_\lambda, \text{ind}_{W_A}^{W_n} \varepsilon_A \rangle &= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \sum_{x \in D_{\hat{\mu}A}} \langle \text{ind}_{W_{x^{-1} \hat{\mu} \cap A}}^{W_A} 1_{x^{-1} \hat{\mu} \cap A}, \varepsilon_A \rangle \\ &= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \sum_{\substack{x \in D_{\hat{\mu}A} \\ x^{-1} \hat{\mu} \cap A = (-1, \dots, -1)}} 1_{x^{-1} \hat{\mu} \cap A} \\ &= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} a_{\hat{\mu}A(-1, \dots, -1)}. \end{aligned}$$

Also,  $\varepsilon(w)$  has the same value for every  $w \in \mathcal{C}(\lambda)$  and so  $\varepsilon(w) = \varepsilon(c_\lambda) = (-1)^{|S_\lambda|}$ . Therefore, by Lemma 3.4, we have

$$\begin{aligned} \langle \xi_\lambda, \text{ind}_{W_A}^{W_n} \varepsilon_A \rangle &= \frac{1}{|W_A|} \sum_{w \in \mathcal{C}(\lambda) \cap W_A} (-1)^{l_A(w^{-1})} \\ &= \frac{1}{|W_A|} \sum_{w \in \mathcal{C}(\lambda) \cap W_A} (-1)^{l(w)} = \frac{1}{|W_A|} (-1)^{|S_\lambda|} |\mathcal{C}(\lambda) \cap W_A| \end{aligned}$$

Putting these two results together, we see that theorem is proved.  $\square$

**Example 3.3.** For  $n = 2$ , let  $S_2 = \{s, t\}$ ,  $S'_2 = \{s, t, sts\}$  and

- $\mathcal{SC}(2) = \{(2), (1, 1), (1, -1), (-1), (-2), (-1, -1)\}$ ;
- $\mathcal{DP}(2) = \{(2), (1, 1), (1, -1), (-2), (-1, -1)\}$ .

The conjugate classes of  $W_2$  are as follows:  $\mathcal{C}((2)) = \{st, ts\}$ ;  $\mathcal{C}((1, 1)) = \{stst\}$ ;  $\mathcal{C}((1, -1)) = \{t, sts\}$ ;  $\mathcal{C}((-2)) = \{s, tst\}$ ;  $\mathcal{C}((-1, -1)) = \{1\}$ . For  $\lambda, \mu \in \mathcal{DP}(2)$

$$(\varphi_\lambda(c_\mu)) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}, \quad (u_{\lambda\mu}) = \begin{pmatrix} 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{-1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}.$$

- $|\mathcal{C}((2))| = |W_2| \cdot \sum_{\mu \in \mathcal{DP}(2)} u_{(2),\mu} = 8 \cdot (1 + \frac{-1}{2} + 0 + \frac{-1}{2} + \frac{1}{4}) = 2$ ;

For  $C = (2), (1, 1), (1, \bar{1}), (\bar{2}), (\bar{1}, \bar{1}) \in \mathcal{DP}(2)$ , the Coxeter generating sets are  $S_C = S_2 = \{s, t\}$ ,  $\{t, stst\}$ ,  $\{t\}$ ,  $\{s\}$ ,  $\emptyset$  respectively. Then we have

$$\begin{aligned} u_{(2),(-1,-1)} &= \frac{(-1)^{|S_2|}}{2 \cdot 2} = \frac{1}{4} \\ u_{(1,1),(-1,-1)} &= (-1)^{|S_{(1,1)}|} \frac{|\mathcal{C}((1, 1))|}{|W_2|} = \frac{1}{8} \\ u_{(1,-1),(-1,-1)} &= (-1)^{|S_{(1,-1)}|} \frac{|\mathcal{C}((1, -1))|}{|W_2|} = \frac{-1}{4} \\ u_{(-2),(-1,-1)} &= (-1)^{|S_{(-2)}|} \frac{|\mathcal{C}((-2))|}{|W_2|} = \frac{-1}{4} \\ u_{(-1,-1),(-1,-1)} &= (-1)^{|S_{(-1,-1)}|} \frac{|\mathcal{C}((-1, -1))|}{|W_2|} = \frac{1}{8}. \end{aligned}$$

Note that these results are the same as from top to bottom entries in the last column of the matrix  $(u_{\lambda,\mu})$ . As considering Corollary 3.1 and Proposition 3.7, we obtain that the set of all  $S_2$ -type elements is  $\mathcal{C}(S_2) = \mathcal{C}((2)) \uplus \mathcal{C}((1, 1))$  and so from (3.4) the number of type  $S_2$  is  $|\mathcal{C}(S_2)| = |\mathcal{C}((2))| + |\mathcal{C}((1, 1))| = 3$ . Also note that here  $|\mathcal{C}(S_2)|$  is equal to the product of exponents of  $W_2$ .

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#### 4- References

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